# Appendix J
# Conditions for Near-Equivalence between $dG^{\dagger}(\nu)/d\nu$ and $\tilde{\alpha}(\nu,\nu)$, and between $d^2G^{\dagger}(\nu)/d\nu^2$ and $d\tilde{\alpha}(\nu,\nu)/d\nu$

Since $\rho = \rho^{\dagger}(\nu)$ marks a stationary point for $G[\rho,\nu]$ where $\partial G / \partial \rho = 0$, it follows from Eq. (13.2) that

$$\left.\begin{aligned}
\frac{dG^{\dagger}(\nu)}{d\nu} &\equiv \frac{dG\left[\rho^{\dagger}(\nu),\nu\right]}{d\nu} = \left(\frac{\partial G[\rho,\nu]}{\partial \nu}\right)_{\rho=\rho^{\dagger}} \\
\frac{d^2G^{\dagger}(\nu)}{d\nu^2} &\equiv \frac{d^2G\left[\rho^{\dagger}(\nu),\nu\right]}{d\nu^2} \doteq \left(-\frac{\partial^2 G[\rho,\nu]}{\partial \rho^2} + \frac{\partial^2 G[\rho,\nu]}{\partial \nu^2}\right)_{\rho=\rho^{\dagger}}
\end{aligned}\right\} \qquad \text{(J-1)}$$

Carrying out this program in Eq. (J-1) and using the asymptotic form for $G[\rho,\nu]$ given in Eq. (5.7-2), we have

$$\left.\begin{aligned}
&\frac{dG^{\dagger}(\nu)}{d\nu} \doteq \\
&-\nu\int_x^{\infty} \frac{d\log n}{d\omega}\frac{d\omega}{\sqrt{\omega^2-\nu^2}} - \pi K_{\nu}\int_{\rho^{\dagger}}^{x} \frac{d\log n}{d\omega}\left(\mathrm{Ai}[\hat{y}]^2 + \mathrm{Bi}[\hat{y}]^2\right)d\omega, \\
&\frac{d^2G^{\dagger}(\nu)}{d\nu^2} \doteq \\
&-\nu\int_x^{\infty} \frac{d^2\log n}{d\omega^2}\frac{d\omega}{\sqrt{\omega^2-\nu^2}} - \pi K_{\nu}\int_{\rho^{\dagger}}^{x} \frac{d^2\log n}{d\omega^2}\left(\mathrm{Ai}[\hat{y}]^2 + \mathrm{Bi}[\hat{y}]^2\right)d\omega
\end{aligned}\right\} \qquad \text{(J-2)}$$

where $\hat{y} \doteq K_\omega^{-1}(\nu - \omega)$ . Here $x$ is a chosen point where the accuracy of the asymptotic forms for the Airy functions is deemed adequate when $\rho \geq x$. From geometric optics, we have from Eqs. (5.6-2) and (5.6-15)

$$\left.\begin{aligned}
&\tilde{\alpha}(\rho,\nu) = -\nu\int_\rho^\infty \frac{d\log n}{d\omega}\frac{d\omega}{\sqrt{\omega^2-\nu^2}}, \ \ \rho \geq \nu, \\
&\frac{d\tilde{\alpha}(\nu,\nu)}{d\nu} = \frac{\tilde{\alpha}}{\nu} - \nu^2\int_\nu^\infty \frac{d}{d\omega}\left(\frac{d\log n}{\omega d\omega}\right)\frac{d\omega}{\sqrt{\omega^2-\nu^2}} \doteq -\nu\int_\nu^\infty \frac{d^2\log n}{d\omega^2}\frac{d\omega}{\sqrt{\omega^2-\nu^2}}
\end{aligned}\right\} \quad \text{(J-3)}$$

Comparison of Eqs. (J-2) and (J-3) yields for $dG\left[\rho^\dagger(\nu),\nu\right]/d\nu$

$$\begin{aligned}
&\frac{dG^\dagger(\nu)}{d\nu} \doteq \\
&\tilde{\alpha}(\nu,\nu) - \left(\tilde{\alpha}(\nu,\nu) - \tilde{\alpha}(x,\nu)\right) - \pi K_\nu \int_{\rho^\dagger}^x \frac{d\log n}{d\rho}\left(\mathrm{Ai}[\hat{y}]^2 + \mathrm{Bi}[\hat{y}]^2\right)d\rho
\end{aligned} \quad \text{(J-4)}$$

From Eq. (5.6-2), we have

$$\tilde{\alpha}(\nu,\nu) - \tilde{\alpha}(x,\nu) = -\nu\int_\nu^x \frac{d\log n}{d\rho}\frac{d\rho}{\sqrt{\rho^2-\nu^2}} \quad \text{(J-5)}$$

Integrating by parts and using Eq. (5.4-3) to express the end value in terms of $\hat{y}$, one obtains

$$\begin{aligned}
&\tilde{\alpha}(\nu,\nu) - \tilde{\alpha}(x,\nu) = \\
&2K_\rho^2\sqrt{-\hat{y}(x,\nu)}\left(\frac{d\log n}{d\rho}\right)_{\rho=x} - \nu\int_\nu^x \frac{d^2\log n}{d\rho^2}\log\left[\frac{\rho+\sqrt{\rho^2-\nu^2}}{\nu}\right]d\rho
\end{aligned} \quad \text{(J-6)}$$

We can continue integrating Eq. (J-6) by parts. It is clear that by successive integrations we can build up a series of terms, all evaluated at $\rho = x$. Similarly, in Eq. (J-4) for $dG\left[\rho^\dagger(\nu),\nu\right]/d\nu$, we have

$$\begin{aligned}
&\pi K_\nu \int_{\rho^\dagger}^{x} \frac{d\log n}{d\rho'}\left(\mathrm{Ai}[\hat{y}]^2+\mathrm{Bi}[\hat{y}]^2\right)d\rho' \doteq \\
&2K_\nu^2\left(\frac{d\log n}{d\rho}\frac{\pi}{2}\int_{\hat{y}(\rho,\nu)}^{\hat{y}^\dagger}\left(\mathrm{Ai}[\hat{y}']^2+\mathrm{Bi}[\hat{y}']^2\right)d\hat{y}'\right)_{\rho=x} \\
&-\pi K_\nu^2 \int_{\rho^\dagger}^{x} \frac{d^2\log n}{d\rho'^2}\left(\int_{\hat{y}^\dagger}^{\hat{y}(\rho',\nu)}\left(\mathrm{Ai}[\hat{y}'']^2+\mathrm{Bi}[\hat{y}'']^2\right)d\hat{y}''\right)d\rho'
\end{aligned} \tag{J-7}$$

It is readily shown that

$$\begin{aligned}
&\frac{\pi}{2}\int_{\hat{y}}^{\hat{y}^\dagger}\left(\mathrm{Ai}[\hat{y}']^2+\mathrm{Bi}[\hat{y}']^2\right)d\hat{y}' = \\
&\frac{\pi}{2}\left(\left(\mathrm{Ai}'[\hat{y}]^2+\mathrm{Bi}'[\hat{y}]^2\right)-\hat{y}\left(\mathrm{Ai}[\hat{y}]^2+\mathrm{Bi}[\hat{y}]^2\right)\right) \underset{\hat{y}\to-\infty}{\longrightarrow} (-\hat{y})^{1/2}
\end{aligned} \tag{J-8}$$

In particular, when $x = \nu + 2K_\nu$,

$$\frac{\pi}{2}\int_{-2}^{\nu^\dagger}\left(\mathrm{Ai}[\hat{y}]^2+\mathrm{Bi}[\hat{y}]^2\right)d\hat{y} = 1.419 \doteq \sqrt{2} \tag{J-9}$$

Thus, even for $x$ as low as $x = \nu + 2K_\nu$, the end terms in Eqs. (J-6) and (J-7) are equal to three significant figures. When $n(\rho)$ is slowly varying, it follows that

$$\tilde{\alpha}(\nu,\nu) - \tilde{\alpha}(x,\nu) + \pi K_\nu \int_{\rho^\dagger}^{x} \frac{d\log n}{d\rho}\left(\mathrm{Ai}[\hat{y}]^2+\mathrm{Bi}[\hat{y}]^2\right)d\rho \doteq 0 \tag{J-10}$$

The accuracy with which Eq. (J-10) holds depends on the curvature in $n(\rho)$, provided that we choose $x > \nu$ so that the asymptotic forms for the Airy functions are not significantly in error. For the examples shown in Figs. 5-4 and 5-5, $K_\nu / H \sim 10^{-3}$, that is, $dn/d\rho$ is slowly varying relative to the range of $\hat{y}$ values ($\sim -2 \le \hat{y} \le 2$) across which the Airy functions make their transition to asymptotic forms. This ratio is generally small for thin atmosphere conditions.

The accuracy of Eq. (J-10) can be checked by comparison of end terms at $\rho = x$ after successive integration by parts in Eqs. (J-6) and (J-7). For example, for the next integration by parts, it can be shown that

$$\begin{aligned}&\frac{\pi}{2}\int_{\hat{y}}^{\hat{y}^{\dagger}}\left(\int_{\hat{y}}^{\hat{y}^{\dagger}}\left(\mathrm{Ai}[\hat{y}'']^2+\mathrm{Bi}[\hat{y}'']^2\right)d\hat{y}''\right)d\hat{y}'\\&=\frac{\pi}{6}\left(\Gamma(\hat{y}^{\dagger})-\Gamma(\hat{y})\right)\underset{\hat{y}\to-\infty}{\longrightarrow}\frac{2}{3}(-\hat{y})^{3/2}+0.195\cdots\end{aligned}\tag{J-11}$$

where $\Gamma(\hat{y})$ has been given in Eq. (4.9-5) and shown in Fig. 5-12. Thus, the difference between Eqs. (J-6) and (J-7) in the end terms after a second integration by parts is about $0.2\nu n''/n$. If $\rho d^2n/d\rho^2 << \alpha$, then a close correspondence between spectral number in wave theory and impact parameter in ray theory should hold. For an exponential refractivity profile in terms of an impact parameter scale height $H_\rho$, the inequality $\rho d^2n/d\rho^2 << \alpha$ translates into the scale height inequality, $H_\rho >> k^{-1}(r_o/\lambda)^{1/3} \approx 0.01\,\text{km}$. However, $H_\rho$ is an impact parameter scale height. It relates to a distance scale height $H_r$ by $H_\rho = (d\rho/dr)H_r \doteq H_r + Nr$. Therefore, a value $H_\rho = 0$ corresponds to a boundary of a locally super-refracting medium; the critical gradient is $dn/dr = -n/r$, or $H_r \approx 1.5\,\text{km}$. Bending angles are no longer defined for $dn/dr < -n/r$ when the tangency point of the corresponding ray lies within such a layer, or even below it if it is too near the lower boundary.

It follows that when $dn/d\rho$ is slowly varying relative to $\hat{y}$ (i.e., the change in refractivity gradient over the Airy function transition width, from an exponential form to a sinusoidal form, $4K_\rho$, is very small), and specifically when a super-refracting medium is avoided, this near-equivalence between $dG/d\nu$ and $\tilde{\alpha}(\nu,\nu)$ holds. We have from Eqs. (J-4) and (J-11)

$$\frac{dG^{\dagger}(\nu)}{d\nu} = \tilde{\alpha}(\nu,\nu) + \mathrm{O}\left[\rho\frac{d^2n}{d\rho^2}\right], \quad \rho^{\dagger} = \nu - \hat{y}^{\dagger}K_\nu \tag{J-12}$$

Similarly, it can be shown from Eqs. (J-1) through (J-11) that

$$\frac{d^2G^{\dagger}(\nu)}{d\nu^2} = \frac{d\tilde{\alpha}(\nu,\nu)}{d\nu} + \mathrm{O}\left[\frac{d^2n}{d\rho^2}\right] \tag{J-13}$$